Section 8.5

Making Rational Functions Simpler

The main idea of this section is to find a way to integrate a rational function like $f(x) = \dfrac{3x}{x^2+2x-8}$. But there is no immediate method (Trig sub can work) to integrate this rational function. If we became aware that f can be written as

$$f(x) = \frac{3x}{x^2+2x-8} = \frac{1}{x-2} + \frac{2}{x+4}$$

then we can quickly integrate f.

$$\int f(x)dx = \int \left(\frac{1}{x-2} + \frac{2}{x+4}\right)dx = \ln|x-2| + 2\ln|x+4| + C.$$

This new form of f is its *partial fraction decomposition.*

To find the partial fraction decomposition we need to do some algebra:

Partial Fractions with Simple Linear Factors

Suppose $f(x) = p(x)/q(x)$, is a proper rational function. If $q(x) = (x - r_1)(x - r_2)\cdots(x - r_n)$ with $r_1, r_2, \ldots, r_n$ all distinct, then

$$\frac{p(x)}{q(x)} = \frac{A_1}{x - r_1} + \frac{A_2}{x - r_2} + \cdots + \frac{A_n}{x - r_n}$$

To find $A_1, A_2, \ldots, A_n$, multiply both sides of the above equation by $q(x)$ and solve for the coefficients A_i be equating like powers.

Example 8.5.1. Find the partial fraction decomposition for $f(x) = \dfrac{3x^2 + 7x - 2}{x^3 - x^2 - 2x}$ then evaluate $\int f\,dx$.

Partial Fractions with Repeated Linear Factors

If the denominator $q(x)$ has a repeated linear factor $(x-r)^m$, then the partial fraction decomposition has a partial fraction for each power of $(x-r)$ up to and including m; that is, it contains

$$\frac{B_1}{(x-r)} + \frac{B_2}{(x-r)^2} + \cdots + \frac{B_m}{(x-r)^m}.$$

The constants B_i are found via the same method as for simple linear factors.

Partial Fractions with Irreducible Quadratic Factors

If the denominator $q(x)$ has a simple irreducible factor ax^2+bx+c, then the partial fraction decomposition has a term of the form

$$\frac{Ax+B}{ax^2+bx+c}.$$

If this factor is repeated, $(ax^2+bx+c)^m$, then the partial fraction decomposition contains a term for each power of (ax^2+bx+c) up to and including m:

$$\frac{A_1x+B_1}{(ax^2+bx+c)} + \frac{A_2x+b_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_mx+B_m}{(ax^2+bx+c)^m}.$$

Example 8.5.2. Use the method of partial fractions to evaluate the following integrals.

1. $\displaystyle\int \frac{x^3-10x^2+27x}{x^2-10x+25}dx$

2. $\displaystyle\int \frac{2}{x^3+x^2}dx$

3. $\int \frac{16x^2}{(x-6)(x+2)^2}dx$

4. $\int \frac{x^4+1}{x^3+9x}dx$

5. $\int \frac{2}{x(x^2+1)^2}dx$